Shock decay in a stress-relaxing fluid in steady plane flow*

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A constitutive relation for a stress-relaxing fluid is combined with equations of motion and continuity for two-dimensional steady plane flow. Coordinates are attached to an arbitrary point on a shock front, and a differential equation is obtained for the decay rate of shock pressure with distance along the shock front. The decay rate depends on the stress-relaxation function, the pressure gradient normal to the shock front, and the curvature of the shock front. Except for the last term, the decay equation is shown to transform to that which applies to plane time-dependent flow. It is suggested that in some cases the relaxation function as a function of pressure can be obtained by measuring the shock-front profile.

The study of shock-wave decay and its relation to stress-relaxation processes in one-dimensional plane shock waves has been fruitful.[1,2] It is appropriate to determine whether or not additional information can be obtained from the study of decay in the next simplest geometry: two-dimensional steady plane flow. This paper gives an equation for the decay of pressure along the shock front which is analogous to that given by Duvall and Ahrens for one-dimensional plane flow.[3]

Consider a shock front, ORQ at time t (Fig. 1), which is being produced by a disturbance at O travelling from right to left at speed q_0. In time dt it has advanced to O′R′Q′ without a change of form. The flow field is unchanging in coordinates fixed in the shock front. The shock front is cylindrical, i.e., it is unchanged by translation in a direction normal to the plane of the paper. Particle velocity is zero ahead of the shock and its direction immediately behind the shock front is normal to the tangent plane. Shock pressure p varies along ORQ and so, consequently, do particle and shock velocity. Every point on the shock front advances in the direction of the normal to its tangent plane.

The coordinates in Fig. 1 have been chosen so that the normal to the tangent plane at R is the x axis, and the y axis lies in the tangent plane and in the plane of the paper. We wish to find the difference between shock pressures at Q′ and R. To do this we define the directional derivative dp/dt to satisfy the relation

$$p(\mathrm{Q}') - p(\mathrm{R}) = dt\frac{dp}{dt}\,.$$

In general, since $p = p(x, y, t)$,

$$dp = p_t\,dt + p_x\,dx + p_y\,dy,$$

where dt, dx, and dy are arbitrary and the subscripts denote differentiation: $p_t = \partial p/\partial t$, etc. In the present case, $y = 0$ at both R and Q′, so $dy = 0$. Moreover, $dx = \mathrm{RQ}' = D\,dt$, where D is shock-propagation velocity at R. Then

$$\frac{dp}{dt} = p_t + Dp_x; \quad \frac{du}{dt} = u_t + Du_x; \quad \frac{dv}{dt} = v_t + Dv_x. \tag{1}$$

For a stress-relaxing material with a constitutive relation

$$\frac{dp}{dt} = c^2\frac{d\rho}{dt} - F(p, \rho),$$

where ρ is density, c is sound velocity, and F is the relaxation function, the equations of two-dimensional irrotational plane flow can be written

$$p_t + up_x + vp_y + \rho c^2(u_x + v_y) = -F, \tag{2}$$

$$u_t + uu_x + vu_y + p_x/\rho = 0, \tag{3}$$

$$v_t + uv_x + vu_y + p_y/\rho = 0, \tag{4}$$

$$u_y - v_x = 0. \tag{5}$$

Equation (2) is the equation of continuity, Eqs. (3) and (4) are the equations of motion, and Eq. (5) is the irrotational condition. Entropy changes are neglected since they complicate the analysis and have but a secondary effect on the decay.[4] Components of flow velocity are denoted by u and v. Equations (2)–(5) apply everywhere except at the shock front.

Now, by combining Eqs. (1)–(4) to eliminate partial time derivatives,

$$\frac{dp}{dt} - (D-u)p_x + vp_y + \rho c^2(u_x + v_y) = -F, \tag{6}$$

$$\frac{du}{dt} - (D-u)u_x + vu_y + \frac{p_x}{\rho} = 0, \tag{7}$$

$$\frac{dv}{dt} - (D-u)v_x + uv_y + \frac{p_y}{\rho} = 0. \tag{8}$$

Combine Eqs. (5)–(8) to eliminate u_x, v_x, and u_y. The remaining equation is

$$(D-u)^2\frac{dp}{dt} + \rho c^2(D-u)\frac{du}{dt} + (D-u)[c^2 - (D-u)^2]p_x$$
$$= -\rho c^2 v\frac{dv}{dt} - v[c^2 + (D-u)^2]p_y - \rho c^2[v^2 + (D-u)^2]v_y$$
$$- (D-u)^2F. \tag{9}$$

Equation (9) is to be evaluated at point R in Fig. 1, immediately behind the shock front. Here $v = 0$ and the pressure p is related to particle velocity, $u_p \equiv u$, by the momentum jump condition

$$p = \rho_0 D u_p.$$

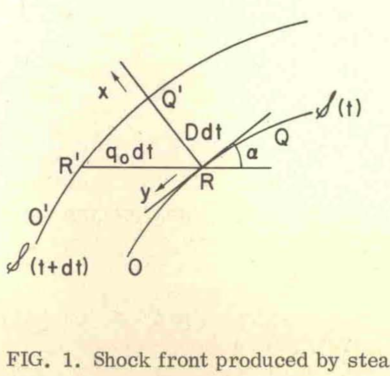

FIG. 1. Shock front produced by steady disturbance at O.

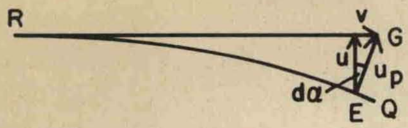

FIG. 2. Relation between v_y and α_y.

We then differentiate to obtain

$$du_p = dp\,(1-\rho_0 u_p D')/\rho_0 D, \tag{10}$$

where

$$D' \equiv \frac{dD}{dp}.$$

Equation (10) can also be written in the form

$$du_p = V\,dp[1+(D-u_p)^2/c'^2][2(D-u_p)]^{-1}, \tag{10'}$$

where V is specific volume and $c'^2 = -V^2\,dp/dV$, and the derivative is taken along the Hugoniot (p, V) curve.

Setting $v=0$ in Eq. (9) and introducing Eq. (10) produces the equation

$$\left(D-u+\frac{c^2(1-\rho_0 u D')}{D-u}\right)\frac{dp}{dt}+[c^2-(D-u)^2]p_x$$
$$=-(D-u)(F+\rho c^2 v_y), \tag{11}$$

where $u \equiv u_p$.

For $v_y=0$, Eq. (11) is identical to the decay equation for one-dimensional plane flow.[3] All two-dimensional effects are incorporated in the v_y term; their net effect is to augment the effect of the relaxation function by the term $\rho c^2 v_y$, which is normally positive, as will be seen later.

For $c=c'$ and $D-u\simeq c$, Eq. (11) becomes

$$\frac{dp}{dt}\simeq-(u+c-D)\,p_x-\tfrac{1}{2}(F+\rho c^2 v_y). \tag{12}$$

The last term on the right-hand side can be evaluated with reference to Fig. 2. RQ is the shock front and EG represents particle velocity u_p at E. u_p can be resolved into components u and v as shown. The angle between u and u_p is $d\alpha=-K\,ds=-ds/r$, where K is curvature of the shock front and $ds=$ RQ. y decreases from R to G. As shown,

$$v(E)\simeq u_p\,d\alpha,$$

$$\mathrm{RG}=-dy,$$

$$dv=v(E)-v(R)=v(E)-0.$$

Therefore,

$$v_y=u_p\alpha_y. \tag{13}$$

Since dD/dp is normally positive, for a decaying shock, $\alpha_y>0$ and $v_y>0$.

Refer again to Fig. 1. The chord R′Q′ $=q_0\,dt\cos\alpha$. Call this $-dy$. Because of the steady-flow assumption, the decay in p going from R′ to Q′ is the same as from R to Q′. Therefore, $dp/dt=-q_0\cos(\alpha)\,p_y$. Substituting this and Eq. (13) into Eq. (11) yields

$$-\left(D-u_p+\frac{c^2(1-\rho_0 u_p D')}{D-u_p}\right)q_0\cos(\alpha)\,p_y+[c^2-(D-u_p)^2]p_x$$
$$=-(D-u_p)(F+\rho c^2 u_p\alpha_y). \tag{14}$$

In the approximation of Eq. (12) this becomes

$$\frac{-\partial p}{\partial y}=\frac{-(u+c-D)p_x}{q_0\cos\alpha}+\frac{F-\rho c^2 u_p\alpha_y}{2q_0\cos\alpha}. \tag{15}$$

If the contour of the shock front is known and if p_x and F are known, Eq. (14) or (15) can be used to calculate the rate of change of p along the shock front. If $p_x=0$ and p_y and α_y are measured, as has been done in explosive wedge experiments,[5] the relaxation function can be estimated from a single experiment. If the Hugoniot of the unrelaxed fluid is known, p_y can be expressed in terms of α_y, so that only the trace of the shock front and p_x need be known in order to determine F. This may provide a useful technique for determining the transition kinetics of shock-induced phase changes.

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